SEMIEMPIRICAL THEORY OF TURBULENT TRANSFER

A. T. Onufriev

A closed system of equations is constructed for flow of a nonisotropic character on the assumption that the mixing path length is not small compared with the characteristic dimension of the stream. It is assumed that the velocity pulsation field can be characterized by a multipoint distribution function, which satisfies the continuity equation. This enables equations to be obtained for the one-point and two-point distribution function. A series of assumptions is made concerning the nature of the forces acting on the turbulent formation ("turbule" or vortex) in the stream and concerning the correlation time of the random force with the scale and intensity of the turbulence. Assumptions are also made concerning the expression of the integral in the equation for the one-point distribution function and the expression for the correlation tensor in the isotropic case. After the moments are calculated, a system of Reynolds' equations is obtained in which approximations, usually acceptable from dimensional considerations, follow for a series of terms. Here, this is a consequence of the approximations for the forces in the equation for the distribution function. Closing the system of equations for the moments reduces to solving the equation for the distribution function. It turns out that the integral character of the transfer (diffusion of a nongradient type) is connected with taking third-order moments into account. A series of examples of flow is considered, and values of the empirical constants are determined. The system of equations obtained enables us to consider flow with strong anisotropy of turbulent transfer.

The commonly known results in the theory of turbulent transfer rely on the equations for the second moments and dimensional considerations for expressing the appropriate terms in the equation of turbulent energy balance by way of the intensity and scale of the turbulence. They are contained in the investigations of A. N. Kolmogorov, L. Prandel, J. Rotta, etc. (a bibliography and review of these papers can be found in [1]).

Attempts have been made in a series of papers to describe the process of turbulent transfer by applying the kinetic equation [2], and also using an analogy with the processes of neutron transfer and radiation [3, 4]; the appropriate bibliography is given in [4].

1. A Model of Turbulent Transfer

It is assumed that the multipoint distribution function which can be used to characterize the velocity pulsation field, $f^{(N)}$ (q_i, p_i, T, χ) (i = 1, ..., N) (q_i are the coordinates, p_i are the momenta of the formations, T is the temperature, and χ is the concentration of additive) satisfies the continuity equation

$$\frac{\partial f^{(N)}}{\partial t}+\frac{\partial}{\partial q_i}\left[\frac{p_i}{m}f^{(N)}\right]+\frac{\partial}{\partial p_i}\left[\frac{dp_i}{dt}f^{(N)}\right]+\frac{\partial}{\partial T_i}\left[\frac{dT}{dt}f^{(N)}\right]+\frac{\partial}{\partial\chi}\left[\frac{d\chi}{dt}f^{(N)}\right]=0 \qquad (1.1)$$

Dolgoprudnyi. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, Vol. 11, No. 2, pp. 62–71, March-April, 1970. Original article submitted July 20, 1969.

(only flow with small variations of temperature and concentration is considered, so that quantities with a fluctuation of density ought to be neglected everywhere, apart from the term with the acceleration of gravity [1].)

The simplest assumption is made as regards the scales of the structures. At each point in the stream the dimension of the turbules is characterized by one value of the scale proportional to the integral scale of the correlation, associated with the length of the mixing path, defined as the distance characterizing the loss of correlation between the initial and final positions of the turbule [5].

The joint distribution coefficient for n points

$$f^{(n)} = \int f^{(N)} \prod_{n+1}^{N} d\mathbf{p}_m \, d\mathbf{q}_m$$

satisfies the equation

$$\frac{\partial f^{(n)}}{\partial t} + \frac{\partial}{\partial q_i}\left[\frac{p_i}{m} f^{(n)}\right] + \int\limits_{1\leqslant i\leqslant n} \frac{\partial}{\partial p_i}\left[\frac{dp_i}{dt} f^{(N)}\right] \prod_{n+1}^{N} d\mathbf{p}_m \, d\mathbf{q}_m + \frac{\partial}{\partial T}\left[\frac{dT}{dt} f^{(n)}\right] + \frac{\partial}{\partial \chi}\left[\frac{d\chi}{dt} f^{(n)}\right] = 0 \tag{1.2}$$

One of the fundamental assumptions concerns the form of the expression for the force acting upon a turbule. It is assumed that this force consists of two parts: the first part $F_i = (dp_i/dt)_1$ describes the hydrodynamic interaction of the turbule with the flow because of the existence of a relative velocity, and has a form similar to that for the force acting upon a sphere of radius L; the second part is associated with the action of the pressure pulsations on the turbule. This is a random force and is a function of all the field coordinates, The method of random forces for describing the field of a turbulent flow was employed by E. A. Novikov in [6]. It is assumed that the pressure pulsations change fairly rapidly compared with the change in the distribution function, and the force associated with their interaction has a correlation time τ. Equation (1.2) can then be integrated within the limits of the time correlation from $-\tau$ to 0 to give [7]

$$\begin{aligned}\frac{\partial f^{(n)}}{\partial t} + \frac{\partial}{\partial q_i}\left[\frac{p_i}{m} f^{(n)}\right] + \frac{\partial}{\partial p_i}\left[\left(\frac{dp_i}{dt}\right)_1 f^{(n)}\right] + \frac{\partial}{\partial T}\left[\frac{dT}{dt} f^{(n)}\right] \\ + \frac{\partial}{\partial \chi}\left[\frac{d\chi}{dt} f^{(n)}\right] = \frac{1}{\tau}\left[-f^{(n)} + \int f^{(n)}(\mathbf{p}_s - \Delta\mathbf{p}_s) W(\Delta\mathbf{p}_s) \Pi d(\Delta\mathbf{p}_s)\right]\end{aligned} \tag{1.3}$$

Here $W(\Delta p)$ is the probability that the momentum will depart from its average value by Δp. Since Eqs. (1.3) cannot be solved exactly we shall attempt to use them to obtain expressions for the one-point distributions as functions of the field parameters for an inhomogeneous stream.

We take $f^{(1)} = f$ for n = 1, and

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial q_i}\left[\frac{p_i}{m} f\right] + \frac{\partial}{\partial p_i}\left[\left(\frac{dp_i}{dt}\right)_1 f\right] + \frac{\partial}{\partial T}\left[\frac{dT}{dt} f\right] + \frac{\partial}{\partial \chi}\left[\frac{d\chi}{dt} f\right] = \frac{1}{\tau}\left[-f + \int f(\mathbf{p} - \Delta\mathbf{p}) W(\Delta\mathbf{p}) d(\Delta\mathbf{p})\right] \tag{1.4}$$

The following approximate expression is used for the integral of the right side:

$$\int f(\mathbf{p} - \Delta\mathbf{p}) W(\Delta\mathbf{p}) d(\Delta\mathbf{p}) = f_0$$

$$f_0 = \langle\rho\rangle \left(\frac{4\pi E}{3}\right)^{-3/2} \exp\left[-\frac{3(u_i - \langle u_i\rangle)^2}{4E}\right] (\pi\langle T'^2\rangle)^{-1/2} \exp\left[-\frac{(T - \langle T\rangle)^2}{\langle T'^2\rangle}\right] \quad (\pi\langle\chi'^2\rangle)^{-1/2} \exp\left[-\frac{(\chi - \langle\chi\rangle)^2}{\langle\chi'^2\rangle}\right],$$

$$E = \frac{1}{2}\langle u_k'^2\rangle$$

Here f_0 is the local-equilibrium value of the distribution function. The considerations leading to this approximation are purely qualitative and are based on the fact that this state of affairs holds in the case of complete equilibrium, and on the fact that this approximation is widely used in the kinetic theory of gases, although the meaning of the right side is different there.

Following the usual practice, we have recourse to dimensional considerations to obtain a relation for the quantity τ,

$$\tau = ALE^{-1/2} \tag{1.5}$$

where A is an empirical constant. Equation (1.4) can then be simplified considerably:

$$\frac{\partial f}{\partial t}+\frac{\partial}{\partial q_i}\left[\frac{p_i}{m}f\right]+\frac{\partial}{\partial p_i}\left[\left(\frac{dp_i}{dt}\right)_1 f\right]+\frac{\partial}{\partial T}\left[\frac{dT}{dt}f\right]+\frac{\partial}{\partial \chi}\left[\frac{d\chi}{dt}f\right]=(f_0-f)/\tau \tag{1.6}$$

Several relations must be assumed for the quantities $F_i = (dp_i/dt)_1$, dT/dt and $d\chi/dt$. They are based on experimental data concerning the processes of friction, heat exchange, and mass exchange for a body moving in a fluid. In particular, we assume

$$F_i = -\frac{1}{\rho}\frac{\partial \langle P\rangle}{\partial x_i} - g\delta(x_i, z) - |C|\, u_i' \frac{3a_0}{8L} \tag{1.7}$$

Here the resistance force is approximated by an expression which is valid for large values of the local Reynolds' number (see also [3, 8]).

It would be desirable to retain the integral character of the solution for the distribution functions f and $f^{(2)}$ completely, but this could be done only for the function f. A complete theory could be developed if a more satisfactory representation of f and $f^{(2)}$ could be obtained.

2. The Equations. Multiplying Equation (1.6) by the quantity $Q(u, T, \chi)$ and integrating completely with respect ot the variables u, T, χ, we obtain the transport equation

$$\frac{\partial}{\partial t}\langle\rho\rangle\langle Q\rangle+\frac{\partial}{\partial x_i}\langle\rho\rangle\langle u_i Q\rangle-\langle\rho\rangle\left\langle F_i\frac{\partial Q}{\partial u_i}\right\rangle-\langle\rho\rangle\left\langle\frac{dT}{dt}\frac{\partial Q}{\partial T}\right\rangle-\langle\rho\rangle\left\langle\frac{d\chi}{dt}\frac{\partial Q}{\partial \chi}\right\rangle=\int\tau^{-1}Q(f_0-f)\,du\,dT\,d\chi \tag{2.1}$$

Assuming that $Q = 1, u_\alpha, u_\alpha^2, u_\alpha u_\beta, T, u_\alpha T, T^2, \chi, u_\alpha\chi, \chi^2$ we can obtain equations for the average quantities and the second moments (we do not sum with respect to α and β). Further, it is assumed that the quantity $\langle\rho\rangle$ is a constant. Strictly speaking, this means that the value of the energy of the pulsating motion is constant in the stream. In considering the whole stream, including regions close to a solid wall or a free boundary, where this condition is violated, we have to introduce the quantity $\langle\rho\rangle$, "the turbulent-state density of the stream", which is associated with the intermittence coefficient, and treat the stream as consisting of two states of medium: laminar and turbulent.

The continuity equation is

$$\frac{\partial\langle u_k\rangle}{\partial x_k}=0 \tag{2.2}$$

The equation of motion for average values of the velocity is

$$\frac{\partial}{\partial t}\langle u_i\rangle+\frac{\partial}{\partial x_k}[\langle u_k\rangle\langle u_i\rangle+\langle u_i'u_k'\rangle]=-\frac{1}{\langle\rho\rangle}\frac{\partial\langle P\rangle}{\partial x_i}-g\delta(x_i, z) \tag{2.3}$$

The equations for components of the Reynolds' stress tensor are

$$\begin{aligned}&\frac{\partial}{\partial t}\langle u_\alpha'u_\beta'\rangle+\frac{\partial}{\partial x_k}[\langle u_k\rangle\langle u_\alpha'u_\beta'\rangle+\langle u_k'u_\alpha'u_\beta'\rangle]+\langle u_\beta'u_k'\rangle\frac{\partial\langle u_\alpha\rangle}{\partial x_k}+\langle u_\alpha'u_k'\rangle\frac{\partial\langle u_\beta\rangle}{\partial x_k}\\&=-g\frac{\langle\rho'u_\beta'\rangle}{\langle\rho\rangle}\delta(x_\alpha, z)-g\frac{\langle\rho'u_\alpha'\rangle}{\langle\rho\rangle}\delta(x_\beta, z)-\langle|C|u_\alpha'u_\beta'\rangle 3a_0/4L+\tau^{-1}[{}^2/_3E\delta(\alpha,\beta)-\langle u_\alpha'u_\beta'\rangle]\end{aligned} \tag{2.4}$$

The energy equation is

$$\frac{\partial C_p\langle T\rangle}{\partial t}+\frac{\partial}{\partial x_k}[C_p\langle T\rangle\langle u_k\rangle+C_p\langle T'u_k'\rangle]=\frac{1}{\langle\rho\rangle}\frac{d\langle P\rangle}{dt} \tag{2.5}$$

The equations for vector components of the turbulent heat flux are

$$\begin{aligned}&\frac{\partial}{\partial t}C_p\langle T'u_i'\rangle+\langle u_i'u_k'\rangle\frac{\partial C_p\langle T\rangle}{\partial x_k}+\frac{\partial}{\partial x_k}[\langle u_k\rangle\langle C_pT'u_i'\rangle]+C_p\langle T'u_k'\rangle\frac{\partial\langle u_i\rangle}{\partial x_k}+\frac{\partial}{\partial x_k}C_p\langle T'u_i'u_k'\rangle\\&=-C_p\langle T'u_i'|C|\rangle\frac{3a_0}{8L}-\frac{1}{\langle\rho\rangle}\frac{d\langle P\rangle}{dt}\frac{\langle\rho'u_i'\rangle}{\langle\rho\rangle}-\tau^{-1}\langle C_pT'u_i'\rangle-C_p\langle T'\rho'\rangle g\delta(x_i, z)\end{aligned} \tag{2.6}$$

The equation for intensity of temperature pulsations is

$$\frac{\partial}{\partial t}\frac{C_p\langle T'^2\rangle}{2}+\frac{C_p}{2}\frac{\partial}{\partial x_k}[\langle u_k\rangle\langle T'^2\rangle+\langle T'^2u_k'\rangle]+C_p\langle T'u_k'\rangle\frac{\partial\langle T\rangle}{\partial x_k}=-\frac{1}{\langle\rho\rangle}\frac{d\langle P\rangle}{dt}\frac{\langle\rho'T'\rangle}{\langle\rho\rangle} \tag{2.7}$$

Equations relating to the concentration field are not written out here. Equations (2.2)-(2.7), (1.6), and (1.5) and the equation for the scale size (which is given in the next section) comprise a closed system. Third-order moments appearing in the equation for the second-order moments must be defined using the solution for the distribution function. We take the expression $\langle | C | u_\alpha' u_\beta' \rangle$ for the quantity $\sim E^{1/2} \langle u_\alpha' u_\beta' \rangle$, etc.

3. The Equation for the Scale L. An approximation equation for L is obtained by integrating the equation for the second-rank correlation tensor and averaging with respect to angle, as was done by Rotta [9]. The difference lies in the fact that the initial equation and approximate expression for the third-order moments in Rotta's paper are obtained on the basis of Equation (1.3) for the distribution function $f^{(2)}$.

This equation is multiplied by $u_\alpha^{(a)} u_\beta^{(b)}$ and integrated over all variable space. The equation of motion at points a and b and the continuity equation are used, and as a result we obtain a transfer equation in the following form. We introduce the following variables: the distance between the points a and b in the stream and the coordinates $x_k^{(ab)}$:

$$\zeta_k = x_k^{(b)} - x_k^{(a)}, \quad x_k^{(ab)} = {}^1/_2\,[x_k^{(b)} + x_k^{(a)}]$$

$$\begin{gathered}
\frac{\partial}{\partial t}\langle u_\alpha'^{(a)} u_\beta'^{(b)}\rangle + \langle u_k'^{(a)} u_\beta'^{(b)}\rangle \left[\frac{\partial \langle u_\alpha\rangle}{\partial x_k}\right]^{(a)} + \langle u_\alpha'^{(a)} u_k'^{(b)}\rangle \left[\frac{\partial \langle u_\beta\rangle}{\partial x_k}\right]^{(b)} \\
+ \frac{1}{2}[\langle u_k^{(a)}\rangle + \langle u_k^{(b)}\rangle]\frac{\partial}{\partial x_k^{(ab)}}\langle u_\alpha'^{(a)} u_\beta'^{(b)}\rangle + [\langle u_k^{(b)}\rangle - \langle u_k^{(a)}\rangle]\frac{\partial}{\partial \zeta_k}\langle u_\alpha'^{(a)} u_\beta'^{(b)}\rangle \\
+ \frac{1}{2}\frac{\partial}{\partial x_k^{(ab)}}[\langle u_\alpha'^{(a)} u_\beta'^{(b)} u_k'^{(b)}\rangle + \langle u_\alpha'^{(a)} u_k'^{(a)} u_\beta'^{(b)}\rangle] \\
+ \frac{\partial}{\partial \zeta_k}[\langle u_\alpha'^{(a)} u_k'^{(b)} u_\beta'^{(b)}\rangle - \langle u_\alpha'^{(a)} u_k'^{(a)} u_\beta'^{(b)}\rangle] \\
= - g\delta(x_\alpha, z)\frac{\langle \rho'^{(a)} u_\beta'^{(b)}\rangle}{\langle\rho\rangle} - g\delta(x_\beta, z)\frac{\langle \rho'^{(b)} u_\alpha'^{(a)}\rangle}{\langle\rho\rangle} - \langle |C|^{(a)} u_\alpha'^{(a)} u_\beta'^{(b)}\rangle \frac{3a_0}{8L_{(a)}} \\
- \langle |C|^{(b)} u_\alpha'^{(a)} u_\beta'^{(b)}\rangle \frac{3a_0}{8L_{(b)}} + \int u_\alpha^{(a)} u_\beta^{(b)} \tau^{-1}(F - f^{(2)})\, du^{(a)}\, du^{(b)}\, dT\, d\chi
\end{gathered} \tag{3.1}$$

Clearly, the scale is not a scalar quantity in the general case. However, in the first approximate treatment involving a whole series of simplifying assumptions (one of which is the introduction of only one average scale at a point), it is convenient to continue considering the simplest case. We thus proceed as follows: we write down the equation for the sum $\langle u_i'^{(a)} u_i'^{(b)}\rangle$, integrated at the point (ab) with respect to the distance ζ between the points a and b, and with respect to angle, to obtain the average values

$${}^1/_2\,[\langle u_k^{(a)}\rangle + \langle u_k^{(b)}\rangle] \approx \langle u_k^{(ab)}\rangle, \qquad L_{(a)} \approx L_{(b)} \approx L_{(ab)}$$

It is assumed that the last term, which depends on pressure pulsations, has the form $\sim\tau^{-1}\langle u_\alpha'^{(a)} u_\beta'^{(b)}\rangle$ and that it can be connected with the terms corresponding to dissipation. In addition, we take

$$\int \langle u_i'^{(a)} u_i'^{(b)}\rangle\, d\zeta\, d\Omega \sim \langle u_i'^2\rangle L = 2LE, \quad \int \langle u_k'^{(a)} u_l'^{(b)}\rangle\, d\zeta\, d\Omega \sim 2L\langle u_k' u_l'\rangle \xi_0$$

As a result, we obtain the following approximate equation for L:

$$\begin{gathered}
\frac{\partial}{\partial t}(LE) + \xi_0 L\langle u_i' u_k'\rangle \frac{\partial\langle u_i\rangle}{\partial x_k} + \langle u_k\rangle \frac{\partial}{\partial x_k}(LE) \\
= -\frac{1}{2}\frac{\partial}{\partial x_k}\left[\int (\langle u_i'^{(a)} u_i'^{(b)} u_k'^{(b)}\rangle + \langle u_i'^{(a)} u_i'^{(b)} u_k'^{(a)}\rangle)\frac{d\zeta\, d\Omega}{4\pi}\right] - \alpha_s E^{3/2} 3a_0/4 - 2\alpha_g g L\langle\rho' u_z'\rangle/\langle\rho\rangle
\end{gathered} \tag{3.2}$$

To determine the functional form of the third-order moments for the inhomogeneous case, we consider the following very simple model: pressure pulsations play the main part in the equation for $f^{(2)}$ and so resistance forces can be neglected. We can then consider the case of the steady-state field when the equation for $f^{(2)}$ in the coordinates $\zeta_k, x_k^{(ab)}$ has the form

$${}^1/_2\,(u_k^{(a)} + u_k^{(b)})\,\partial f^{(2)}/\partial x_k^{(ab)} + (u_k^{(a)} - u_k^{(b)})\,\partial f^{(2)}/\partial\zeta_k = \tau^{-1}(F - f^{(2)}) \tag{3.3}$$

In the homogeneous case $f^{(2)}$ depends only on ζ and satisfies the equation

$$(u_k^{(a)} - u_k^{(b)})\,\partial f_0^{(2)}/\partial\zeta_k = \tau^{-1}(F - f_0^{(2)})$$

A correction to the expression for the homogeneous case is obtained by inserting the value of $f^{(2)}$ in the expression for $f_0^{(2)}$ from (3.3)

$$f^{(2)} = F - \tau (u_k^{(a)} - u_k^{(b)}) \partial f^{(2)} / \partial \zeta_k - {}^1/_2 \tau (u_k^{(a)} + u_k^{(b)}) \partial f^{(2)} / \partial x_k^{(ab)} \approx f_0^{(2)} - {}^1/_2 \tau (u_k^{(a)} + u_k^{(b)}) \partial f_0^{(2)} / \partial x_k^{(ab)} \tag{3.4}$$

The parameters E, L, $\langle u_m \rangle$ in this expression must now be taken as functions of the coordinates. On differentiation the derivatives

$$\partial E / \partial x_k, \quad \partial L / \partial x_k, \quad u_m \partial \langle u_m \rangle / \partial x_k$$

can appear on the right-hand side.

If it is assumed that the fourth- and fifth-order "semi-invariants" (cumulants) [1] are equal to zero, the fourth moments can be expressed in terms of the second, while the fifth moments, which turn out to be proportional to the third in the approximation of weak inhomogeneity under consideration, can be neglected. In particular, we can write

$$\langle u_i'^{(a)} u_i'^{(b)} u_k'^{(b)} \rangle \approx \langle u_i'^{(a)} u_i'^{(b)} u_k'^{(b)} \rangle_0 - \sum_l [\gamma_1' L E^{-3/2} \langle u_i'^{(a)} u_i'^{(b)} u_k'^{(b)} u_l'^{(\ )} \rangle \partial E / \partial x_l$$

$$+ \gamma_2' E^{-1/2} \langle u_i'^{(a)} u_i'^{(b)} u_k'^{(b)} u_l'^{(\ )} \rangle \partial L / \partial x_l + \gamma_3' L E^{-3/2} \langle u_i'^{(a)} u_i'^{(b)} u_k'^{(b)} u_m'^{(ab)} u_l'^{(\ )} \rangle \frac{\partial \langle u_m' \rangle}{\partial x_l} \Big]$$

$$\approx \langle u_i'^{(a)} u_i'^{(b)} u_k'^{(b)} \rangle_0 - \sum_l E^{-3/2} [\langle u_i'^{(a)} u_i'^{(b)} \rangle \langle u_k'^{(b)} u_l'^{(\ )} \rangle$$

$$\langle u_i'^{(a)} u_k'^{(b)} \rangle \langle u_i'^{(b)} u_l'^{(\ )} \rangle + \langle u_i'^{(a)} u_l'^{(\ )} \rangle \langle u_i'^{(b)} u_k'^{(b)} \rangle] \times [\gamma_1' L \partial E / \partial x_l + \gamma_2' E \partial L / \partial x_l]$$

After integrating with respect to angle and distance between the points, we obtain

$$\int \langle u_i'^{(a)} u_i'^{(b)} u_k'^{(b)} \rangle \frac{d\zeta d\Omega}{4\pi} \approx - L E^{1/2} \left[\frac{\langle u_k' u_l' \rangle}{E} + \frac{\langle u_i' u_k' \rangle \langle u_i' u_l' \rangle}{E^2} \right] \left[\gamma_1 L \frac{\partial E}{\partial x_l} + \gamma_2 E \frac{\partial L}{\partial x_l} \right] \tag{3.5}$$

The magnitude of the constants γ_1 and γ_2 must be determined from experimental data. Inserting Eq. (3.5) in (3.2), we obtain the final equation for the scale

$$\frac{\partial}{\partial t} LE + \xi_0 L \langle u_i' u_k' \rangle \frac{\partial \langle u_i \rangle}{\partial x_k} + \langle u_k \rangle \frac{\partial}{\partial x_k} LE = \frac{\partial}{\partial x_k} \Big\{ L E^{1/2} \left[\frac{\langle u_k' u_l' \rangle}{E} + \frac{\langle u_i' u_k' \rangle \langle u_i' u_l' \rangle}{E^2} \right]$$

$$[\gamma_1 L \partial E / \partial x_l + \gamma_2 E \partial L / \partial x_l] \Big\} - \alpha_s E^{3/2} 3 a_0 / 4 - 2 \alpha_g g L \langle \rho' u_z' \rangle / \langle \rho \rangle \tag{3.6}$$

This equation agrees with the equation for the scale obtained in Rotta's paper [9], which involved a series of relations and hypotheses for the spectral functions in its derivation.

4. The equations given above refer to developed turbulent flow and do not allow for processes connected with molecular diffusion. A treatment of flow in the region where the effect of viscosity is important requires that third moments should be correctly allowed for, since they play a deciding part in this region.

The magnitudes of the third moments must be determined from the solution of the equation for the distribution function. A simple form of the solution for f cannot be obtained. We shall thus construct an approximate solution which correctly allows for the fundamental characteristic of the turbulent transfer process: the generation of turbules and their propagantion over large distances. The effect of resistance forces is neglected in the equation for the distribution function although the corresponding terms are, generally speaking, not small, and plane flow is considered in which the flow parameters are functions of the transverse coordinate only. In this case,

$$u_y' \frac{\partial f}{\partial y} = \tau^{-1} (f_0 - f)$$

$$f(y, u_y' < 0) = \int_y^\infty f_0 \exp \left[- \int_y^s \frac{ds}{\tau |u_y'|} \right] \frac{ds}{\tau |u_y'|} \tag{4.1}$$

$$f(y, u_y' > 0) = \int_{-\infty}^y f_0 \exp \left[- \int_s^y \frac{ds}{\tau u_y'} \right] \frac{ds}{\tau u_y'}$$

The equation for the frictional stress has the form

$$\frac{d}{dy} \langle u_x' u_y'^2 \rangle + \langle u_y'^2 \rangle \frac{d \langle u_x \rangle}{dy} = - \tau^{-1} \langle u_x' u_y' \rangle \tag{4.2}$$

When the quantities

$$\langle u_x' u_y' \rangle = \int (u_x - \langle u_x \rangle)\, u_y f\, du_x du_y du_z$$

$$\langle u_y'^2 \rangle = \int u_y^2 f\, du_x du_y du_z \qquad (4.3)$$

$$\langle u_x' u_y'^2 \rangle = \int (u_x - \langle u_x \rangle)\, u_y^2 f\, du_x du_y du_z$$

have been calculated, we can make a direct substitution to check that Eq. (4.2) is satisfied. However, Eq. (4.3) for the frictional stress describes integral transfer for large values of the mixing path, and consequently, diffusion of the nongradient type is equivalent to taking third-order moments into account.

We also note that the approximations for the forces acting upon a turbule, assumed in the equation for the distribution function, led to expressions which were applied for components containing pressure pulsations and were introduced from dimensional and physical considerations. However, there was a difference in the term characterizing energy dissipation. Usually the isotropic expression is used:

$$^3/_4 a_0 L^{-1} E^{3/2} \delta(\alpha, \beta)$$

Here, however, the expression

$$^3/_4 a_0 L^{-1} E^{1/2} \langle u_\alpha' u_\beta' \rangle$$

was obtained, i.e., energy dissipation for components of the kinetic energy of the pulsating motion, for example $\langle u_x'^2 \rangle$, is proportional to the quantity $\sim E^{1/2} \langle u_x'^2 \rangle$, and is not taken to be one third of the over-all magnitude of energy dissipation.

5. The Plane Steady-State Flow of a Turbulent Stream.

We shall consider a plane, steady-state, isothermal stress once more for the case in which the stream parameters vary in the direction of the y axis only. The velocity of the stream U is in the direction of the x axis.

A Layer of Constant Frictional Stress. The flow is considered far from the wall (the effect of viscosity can be neglected). The third-order moments are zero. We then introduce the dimensionless quantities

$$U^+ = U v_*^{-1}, \quad y^+ = y v_* \nu^{-1}$$

As a result, we obtain ($B = 1 + 3/4 a_0 A$)

$$\langle u_x' u_y' \rangle v_*^{-2} = -1$$

$$-2 \frac{dU^+}{dy^+} = -\frac{B \langle u_x'^2 \rangle^+ (E^+)^{1/2}}{AL^+} + {}^2/_3 \frac{(E^+)^{3/2}}{AL^+}$$

$$\frac{B \langle u_y'^2 \rangle^+ (E^+)^{1/2}}{AL^+} = {}^2/_3 \frac{(E^+)^{3/2}}{AL^+}$$

$$\frac{dU^+}{dy^+} = {}^3/_4 \frac{a_0 (E^+)^{3/2}}{L^+}, \qquad \langle u_y'^2 \rangle^+ \frac{dU^+}{dy^+} = \frac{B (E^+)^{1/2}}{AL^+}$$

$$-\xi_1 L^+ \frac{dU^+}{dy^+} = \gamma_2 \frac{d}{dy^+} \left\{ L^+ (E^+)^{3/2} \left[\frac{\langle u_y'^2 \rangle^+}{E^+} \left(1 + \frac{\langle u_y'^2 \rangle^+}{E^+}\right) + (E^+)^{-2} \right] \frac{dL^+}{dy^+} \right\} - \frac{3}{4} \alpha_s a_0 (E^+)^{3/2}$$

These equations give

$$L^+ \frac{dU^+}{dy^+} = \frac{3a_0 (E^+)^{3/2}}{4}, \qquad \frac{\langle u_x'^2 \rangle^+}{E^+} = \frac{2(3B-2)}{3B}$$

$$\frac{\langle u_y'^2 \rangle^+}{E^+} = \frac{2}{3B}, \quad E^+ = B \left[\frac{2}{a_0 A} \right]^{1/2}$$

$$\frac{d}{dy^+} \left(L^+ \frac{dL^+}{dy^+} \right) = \text{const} = \beta_0^2$$

From the last relation, we have

$$L^+ = \beta_0 y^+$$

and from the first

$$U^+ = C_0 + \frac{3a_0B^{3/2}}{4\beta_0}\left(\frac{2}{a_0A}\right)^{3/4}\ln y^+ = C_0 + \varkappa^{-1}\ln y^+$$

i.e., a logarithmic law for the velocity distribution, well known in this case. The value of the constant is $\varkappa = 0.40$. We also take $\beta_0 = 0.40$, $a_0 = 0.345$. Then $a_0A = 4/3$ and $A = 3.86$. For these values of the constants, the relations given above lead to

$$E^+ = 2{,}44 \quad \langle u_x'^2\rangle = 3{,}24, \quad \langle u_y'^2\rangle = 0{,}81$$

These values are in satisfactory agreement with experimental data [10].

Isotropic Plane Flow behind a Grid. For isotropic flow the equations can be simplified considerably since

$$\langle u_x'^2\rangle = \langle u_y'^2\rangle = \langle u_z'^2\rangle = {}^2/_3 E$$

Viscous forces can be neglected, the velocity U is constant, and we have the equations

$$u\frac{dE}{dx} = -\frac{E^{3/2}3a_0}{4L}, \quad u\frac{dLE}{dx} = -\frac{\alpha_s E^{3/2}3a_0}{4}$$

Here third-order moments are also neglected. The relation for the invariant of L. G. Loitsyanskii [11]

$$\Lambda = EL^{1/(1-\alpha_s)}$$

follows from the equations, and so $\alpha_s = 0.8$ since $1 - \alpha_s = 1/5$.

The functions

$$L \sim x^{2/7}, \quad E \sim x^{-10/7}$$

can then be found easily.

This corresponds to the familiar results of A. N. Kolmogorov.

Flow in an Aerodynamic Pipe. In this case, it is assumed that the scale L is constant and proportional to the dimensions of the grid divisions. Thus, the equation [12]

$$\frac{dE}{dt} = -\frac{3}{2}\left(\frac{2}{3}\right)^{3/2}\frac{A''}{l}E^{3/2}, \qquad A'' = 1.0-1.2$$

is valid.

Here l is the longitudinal integral scale, i.e., a quantity proportional to L (close to AL since this value appears in the index of the exponent). In this case, the equation for E is

$$dE / dt = -{}^3/_4\, a_0 L^{-1} E^{3/2}$$

Comparing these two relations, we have the estimate

$$l \approx 4L / 3a_0 = AL$$

which corresponds with the value assumed previously, $3/4\, a_0 A = 1$.

6. Passage to the Limiting Relations of the Mixing-Path Theory

The system of equations describes turbulent transfer without assuming that the size of the mixing path L is small. Mixing-path theory was developed by L. Prandtl on the basis of similarity with molecular transfer in the continuous medium regime when L is small compared with the characteristic dimension of the problem. We arrive at the model of mixing-path theory if we assume that for $L \to 0$ the quantities τ and $\langle u_i'u_j'\rangle$ $(i \neq j)$ are small first-order quantities, while E has a finite value. In order for these conditions to be satisfied, we must assume that $a_0 \to 0$. All these conditions are inconsistent, since in turbulent flow the quantity $\langle u_x'u_y'\rangle$, is not small compared with E, for example.

Under these conditions, the initial equations (not allowing for viscosity) give

$$\langle u_\alpha'^2 \rangle = {}^2/_3 E - 2\nu' \partial \langle u_\alpha \rangle / \partial x_\alpha$$
$$\langle u_\alpha' u_\beta' \rangle = -\nu' (\partial \langle u_\alpha \rangle / \partial x_\beta + \partial \langle u_\beta \rangle / \partial x_\alpha)$$
$$c_p \langle T' u_i' \rangle = -k' \partial c_p \langle T \rangle / \partial x_i$$
$$\nu' = k' = {}^2/_3 A L E^{1/2}, \quad P' = \nu' / k' = 1$$

The expression for the coefficient for turbulent viscosity then becomes a scalar.

If we do not make the limiting transition $L \to 0$, we obtain an expression for the turbulent Prandtl number P' which is different from unity. We treat the particular case of plane-parallel motion (without allowing for viscous forces and third-order moments): ·

$$\langle u_x' u_y' \rangle = -\frac{AL}{(1 + 3a_0 A / 4) E^{1/2}} \langle u_y'^2 \rangle \frac{du}{dy}$$
$$c_p \langle T' u_y' \rangle = -\frac{AL}{(1 + 3a_0 A / 8) E^{1/2}} \langle u_y'^2 \rangle \frac{dc_p \langle T \rangle}{dy}$$

and we have in this case

$$P' = \frac{1 + {}^3/_8 a_0 A}{1 + {}^3/_4 a_0 A} = {}^3/_4$$

for the values assumed for the constants.

The constants a_0, A, α_s, ξ_0, γ_1, γ_2 appeared in the equations. The values of α_s and A are determined from the relationships following from the equations, as we saw above, while the remainder come from comparison with experimental data.

These very simple examples show us that the system of equations obtained enables us to describe the familiar laws, and allows us to consider nongradient-type diffusion and the nonisotropic behavior of components of the kinetic energy of the pulsating motion. It also enables us to introduce nonisotropic turbulent viscosity which turns out to be important in a series of problems concerning turbulent flow in a stratified medium.

LITERATURE CITED

1. A. S. Monin and A. M. Yaglom, Statistical Hydromechanics [in Russian], Part 1, Nauka, Moscow, 1965.
2. G. B. Shubauer and C. M. Tchen, Turbulent Flow [Russian translation], Izd-vo Inostr. Lit., Moscow, 1963.
3. N. I. Buleev, "A theoretical model for the turbulent exchange mechanism in the fluid flow," in: Heat Transfer [in Russian], Izd-vo AN SSSR, Moscow, 1962.
4. A. T. Onufriev, "The equations of integral diffusion. Application to investigating the process of turbulent transfer," PMTF, 8, No. 4, 1967.
5. J. Taylor, "The present state of turbulent diffusion theory," in: Atmospheric Diffusion and Air Pollution [Russian translation], Izd-vo Inostr. lit., Moscow, pp. 129-137, 1962.
6. E. A. Nivokov, "The method of random forces in turbulence theory," ZhÉTF, 44, No. 6, 1963.
7. R. K. Eisenschutz, Statistical Theory of Irreversible Processes [Russian translation], Izd-vo inostr. lit., Moscow, 1963.
8. F. H. Harlow and P. I. Nakayama, "Turbulence transport equations," Phys. Fluids, 10, No. 11, 1967.
9. I. Rotta, "Statistische Theorie nichthomogener Turbulenz," Z. Phys., 129, No. 5, pp. 547-572; 131, No. 1, pp. 51-77, 1951.
10. A. A. Townsend, The Structure of Turbulent Shear Flow [Russian translation], Izd-vo inostr. lit., Moscow, 1969.
11. L. G. Loitsyanskii, "Some fundamental laws of isotropic turbulent flow," Proceedings of the Central Aerohydrodynamics Institute, No. 440, Moscow, 1939.
12. G. Batchelor, Theory of Homogeneous Turbulence [Russian translation], Izd-vo inostr. lit., Moscow, pp. 105-108, 1955.